

# Inconsistency of Breathing Mode Extensions of Maximal Five-Dimensional Supergravity Embedding

James T. Liu<sup>†</sup> and C.N. Pope<sup>‡,\*</sup>

<sup>†</sup>*Michigan Center for Theoretical Physics*

*Randall Laboratory of Physics, The University of Michigan*

*Ann Arbor, MI 48109-1040, USA*

<sup>‡</sup>*George P. & Cynthia W. Mitchell Institute for Fundamental Physics and Astronomy*

*Texas A&M University, College Station, TX 77843-4242, USA*

<sup>\*</sup>*DAMTP, Centre for Mathematical Sciences, Cambridge University,*

*Wilberforce Road, Cambridge CB3 0WA, UK*

## ABSTRACT

Recent work on consistent Kaluza-Klein reductions on Einstein-Sasaki spaces prompted an intriguing conjecture that there might exist a consistent  $S^5$  reduction of type IIB supergravity to give five-dimensional  $\mathcal{N} = 8$  gauged supergravity coupled to a massive supermultiplet that includes the breathing-mode scalar. Motivated by this, we investigate the possibility of augmenting the usual  $\mathcal{N} = 8$  supergravity reduction to include a breathing-mode scalar, and we show that this is in fact inconsistent. The standard reduction to the massless  $\mathcal{N} = 8$  supermultiplet depends for its consistency on a delicate interplay between properties of the ten-dimensional type IIB theory and properties of the Killing vectors on  $S^5$ . Our calculations show that turning on the breathing-mode is sufficient to destroy the balance, and hence render the reduction inconsistent.

# 1 Introduction

Kaluza-Klein reductions provide a mechanism for obtaining lower-dimensional theories from higher-dimensional ones. The simplest example is the reduction of a  $(D + 1)$ -dimensional theory to  $D$  dimensions, by taking the extra coordinate  $z$  to lie on a circle. If one expands the  $z$ -dependence of all higher-dimensional fields in terms of modes on the circle, then one is effectively just Fourier transforming the original higher-dimensional theory. However, in this example one can additionally perform a *consistent truncation*, in which all the infinite towers of massive fields, associated with non-trivial  $z$ -dependent modes on the circle, are set to zero. The crucial point about this truncation is that setting the massive fields to zero is consistent with their own equations of motion. As a result, the remaining lower-dimensional theory, in which only the fields associated with  $z$ -independent modes on the circle are retained, represents a consistent embedding within the original theory. In other words, any solution of the truncated lower-dimensional theory lifts back as a solution in the higher dimension.

There is a simple group-theoretic way of seeing the consistency of the truncation described above. The truncation retains all the singlets, and only the singlets, under the action of the  $U(1)$  isometry group of the circle. For example, a higher-dimensional field  $\Phi(x^\mu, z)$  can be expanded as

$$\Phi(x^\mu, z) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) e^{inz/L}, \quad (1.1)$$

where  $L$  is the radius of the circle. The singlet field,  $\phi_0$ , is uncharged, while the  $\phi_n$  fields with  $n \neq 0$  occur in  $\phi_{\pm n}$  charged doublets. Clearly products of uncharged fields can never act as sources for charged fields, and so setting all the charged fields to zero will necessarily be a solution of their equations of motion.

The above argument can be generalised to the situation where a higher-dimensional theory is reduced on a manifold  $M$  with isometry group  $G$ . If the full infinite towers of modes in a generalised Fourier expansion are retained, then the resulting lower-dimensional theory is necessarily a consistent embedding within the higher-dimensional one. These modes will carry representations of  $G$ . If one truncates the full system so as to keep all the singlets and only the singlets under some subgroup  $K \subset G$ , then this truncation will be guaranteed to be a consistent one. The reason is the same as in the previous circle-reduction example; non-linear terms built from the  $K$ -singlets cannot act as sources for the truncated non-singlet modes, and so setting all the non-singlets to zero is an exact solution of their equations of motion.

A further point is that if the subgroup  $K$  of the isometry group acts transitively on  $M$ , then the number of  $K$ -singlets will be finite. This is generally what is wanted in a consistent truncation. Typically, these singlet modes will include some (but perhaps not all) of the massless fields in the full Kaluza-Klein towers, together, possibly, with some massive fields.

A completely different kind of consistent truncation arises in some special situations, typically, but not always, associated with certain sphere reductions of supergravity theories. The most well-known example is the  $S^7$  reduction of eleven-dimensional supergravity. There exists an (almost cast-iron) proof that it is consistent to truncate the infinite towers of Kaluza-Klein modes so as to retain precisely the fields of four-dimensional  $\mathcal{N} = 8$   $SO(8)$  gauged supergravity [1]. There is no known group-theoretic reason why this truncation is consistent; it depends for its success on very remarkable and delicate conspiracies between properties of eleven-dimensional supergravity and properties of the seven-sphere. Other examples of these “non-trivial” consistent reductions include the  $S^4$  reduction of eleven-dimensional supergravity (for which there exists a complete proof of the consistency [2, 3]), and the  $S^5$  reduction of type IIB supergravity (where there are strong indications of the consistency [4, 5, 6, 7]).

One way of seeing how the properties of the supergravity theory and the sphere conspire to make possible a consistent truncation was exhibited for the  $S^7$  reduction in [8]. The argument focuses on a specific subset of the interactions in the full reduction procedure, namely those associated with the back reaction of the  $SO(8)$  gauge fields on gravity in the lower dimension. Prior to making any truncation, the four-dimensional fields will include a tower of massive spin-2 fields, with the massless graviton at the bottom. These fields arise from the mode expansion of the four-dimensional components of the eleven-dimensional metric, in which the mode functions on the  $S^7$  are the eigenfunctions of the scalar Laplacian. *A priori*, one might expect that quadratic products of the  $SO(8)$  gauge bosons would contain not only a singlet mode (*i.e.* the usual Yang-Mills energy-momentum tensor) which would act as a source for the massless graviton, but also a non-singlet contribution that would act as a source for massive spin-2 fields. This would prevent one from consistently setting the massive spin-2 fields to zero.

In fact, this is exactly what would happen if a *generic* theory of gravity were reduced on a sphere. The special features of eleven-dimensional supergravity and the seven-sphere that save the day are that the quadratic product of gauge bosons couples to spin-2 fields through two completely different sources, one being via the Kaluza-Klein metric reduction ansatz, and the other via the reduction ansatz for the 4-form field strength of eleven-dimensional

supergravity. When the two contributions are taken together, the non-singlet terms in the quadratic product of gauge fields cancel out, and so the gauge bosons excite only the (retained) massless graviton, but not the (truncated) massive spin-2 fields.

This is only a *necessary* condition for the full consistency of the truncation to the massless sector, but it is a very revealing one. In fact the only known cases where reductions pass this test are precisely those where in fact a fully-consistent reduction is either known to exist, or strongly believed to exist.

In this paper, we shall make use of an extension of the consistency test in [8] in order to study an intriguing recent conjecture that even more remarkable consistent truncations might be possible [9, 12]. This suggestion has arisen as a result of recent constructions of certain consistent reductions of eleven-dimensional supergravity on Einstein-Sasaki 7-manifolds [9], and of type IIB supergravity on Einstein-Sasaki 5-manifolds [10, 11, 12, 13]. These reductions involve a consistent truncation to a set of modes that includes massive as well as massless fields.

Taking the type IIB reductions as an example, the new consistent truncation can be stated most straightforwardly in the case that the compactifying Einstein-Sasaki space  $M_5$  is simply  $S^5$ . The isometry group of  $S^5$  is  $SO(6)$ , but this contains an  $SU(3)$  subgroup that acts transitively on  $S^5$ . This is easily seen from the fact that  $S^5$  can be viewed as a  $U(1)$  bundle over  $CP^2$ , and this admits a natural action by isometries of  $SU(3) \times U(1)$ , where  $SU(3)$  acts on  $CP^2$  and  $U(1)$  acts on the fibres. The statement of the new consistent reduction is that one retains all the singlets, and only the singlets, under  $SU(3)$ . The truncation is therefore guaranteed to be a consistent one. Note that included in this truncation are two massive scalar modes, associated with an overall “breathing” deformation of the  $S^5$ , and a “squashing” of the  $U(1)$  fibres relative to the  $CP^2$  base [14].

Although the consistency of this reduction is obvious when the compactifying space  $M_5$  is just  $S^5$ , it is less obvious, though nevertheless true, that  $S^5$  can be replaced by *any* five-dimensional Einstein-Sasaki space, with essentially the same reduction ansatz, and consistency is again achieved. (One can describe any Einstein-Sasaki space as a  $U(1)$  bundle over an Einstein-Kähler base, and the modes analogous to those that are retained in the  $S^5$  reduction can be constructed in an almost identical fashion when  $CP^2$  is replaced by any other Einstein-Kähler base, since only the invariant Kähler form and holomorphic 2-form play a rôle.)

Motivated by these recently-constructed consistent reductions, it has been suggested in [9, 12] that it might be possible to augment the full  $\mathcal{N} = 8$  non-trivial consistent reduction

of type IIB supergravity on  $S^5$  by including in addition certain massive supermultiplets. In particular, one of these massive supermultiplets would include the breathing-mode scalar. There would also be associated massive spin-2 fields in the supermultiplet.

There are various reasons why one might be sceptical about this possibility. First of all, it would violate a widely-held belief that it is not possible to have a consistent coupling of (a finite number of) massive spin-2 fields to gravity. (Kaluza-Klein theories, with the complete towers of massive fields, do, of course, routinely manage to couple massive spin-2 to gravity. But in such examples, it is an *infinite* number of massive spin-2 fields, and this appears to provide the loophole to the standard lore.)

A related observation is that the massive spin-2 fields in the Kaluza-Klein towers are associated with scalar harmonics on  $S^5$ . The massive spin-2 fields contained in the supermultiplet that includes the breathing mode lie in the **20** representation of  $SO(6)$ . *A priori*, one would expect that there would be cubic terms in the complete lower dimensional Lagrangian, describing the coupling of two of these massive spin-2 fields in the **20** representation to a third spin-2 field, where this third field could be in any of the  $SO(6)$  representations for scalar harmonics on  $S^5$  that could arise in the symmetric product of the two **20** representations, namely the **1**, **20** and the **105**. In the first two cases, these cubic interactions would be associated with the **20** massive spin-2 fields giving energy-momentum tensor contributions sourcing the massless graviton and the **20** of massive spin-2 fields themselves. Both of these cubic interactions would be perfectly consistent with the conjecture that the breathing-mode multiplet could be retained in a consistent truncation. The third of the cubic interactions listed above, however, would correspond to the **20** massive spin-2 fields sourcing the yet-higher mass **105** of spin-2 fields. This would then be the start of an infinite sequence of interactions that would be inconsistent with having set the **105** of massive spin-2 fields, and those of yet higher mass, to zero. Since there is only one term in the ten-dimensional theory that contributes to this class of cubic interactions, there is no possibility of any delicate cancellation between contributions that might conspire to project out the **105** representation in the harmonic expansion. The situation for these cubic interactions for massive spin-2 fields is therefore quite different from the situation for the gauge bosons in the  $S^5$  or  $S^7$  supergravity reductions described earlier, where the fact that they enter in the higher-dimensional equations from two distinctly different sources (the metric and the antisymmetric tensor) provided the opportunity for a delicate cancellation that avoided the otherwise expected excitation of massive spin-2 fields.

A further reason for scepticism is that evidence coming from the recently-constructed

reductions that include massive fields is actually not very persuasive. As we noted above, in the case where the internal space is  $S^5$ , the consistency of the existing  $\mathcal{N} = 2$  reduction with the breathing-mode multiplet is in fact guaranteed by group-theoretic considerations, and so does not provide any non-trivial test of the kind of conspiracies that would be necessary if a reduction to  $\mathcal{N} = 8$  gauged supergravity plus the breathing-mode multiplet were to be consistent.

In this paper, we shall show by considering a specific class of interactions within a hypothetical  $S^5$  reduction to the massless supergravity multiplet plus breathing-mode multiplet that conditions necessary for the consistency of the truncation are not fulfilled. Specifically, we shall show that the inclusion of the breathing mode is sufficient to destroy the delicate conspiracies that allowed the truncation purely to the massless  $\mathcal{N} = 8$  sector to work. Supplementing the massless sector and the breathing mode by the rest of the associated massive supermultiplet would not aid matters. Thus, we conclude that it is not possible to obtain a consistent truncation which includes the breathing mode as well as the usual massless modes of  $\mathcal{N} = 8$  gauged  $SO(6)$  supergravity.

Although our analysis in this paper focuses on the  $S^5$  reduction of type IIB supergravity, analogous arguments should hold in other cases too, such as the  $S^7$  reduction of eleven-dimensional supergravity. Thus, it is to be expected that turning on the breathing mode in the reduction of eleven-dimensional supergravity to four-dimensional  $\mathcal{N} = 8$  gauged  $SO(8)$  supergravity will similarly destroy the consistency of the reduction.

## 2 Breathing Mode Consistency Condition

As demonstrated in [8] for  $D = 11$  supergravity and [15, 16] for IIB supergravity, the manner in which the Kaluza-Klein gauge fields back-react on the lower dimensional metric provides a rather non-trivial consistency condition on the reduction. The complete ansatz for the consistent  $S^5$  reduction of the gravity plus self-dual 5-form sector of type IIB supergravity, yielding gravity coupled to  $SO(6)$  Yang-Mills with 20 scalar fields, was obtained in [7].

Here, we shall extend the analysis of the consistency condition on the IIB supergravity reduction, by augmenting the discussion of [15, 16] to include a breathing-mode scalar. The relevant fields of IIB supergravity are the metric  $\hat{G}_{MN}$  and the self-dual five-form field strength  $\hat{H}_5$ . The IIB equations of motion for these fields are

$$\begin{aligned}\hat{R}_{MN} &= \frac{1}{96}\hat{H}_{MPQRS}\hat{H}_N{}^{PQRS}, \\ d\hat{H}_5 &= 0, \quad \hat{H} = \hat{*}H_5.\end{aligned}\tag{2.1}$$

(We use hats to denote 10-dimensional quantities.) In order to demonstrate a necessary condition for consistency, we may work at the linearized level for the gauge bosons but include the breathing mode in a non-linear manner. We thus take the ansatz

$$\begin{aligned} d\hat{s}^2 &= e^{5\phi} e^\alpha e^\beta \eta_{\alpha\beta} + e^{-3\phi} (e^a - K^{Ia} A^I) (e^b - K^{Jb} A^J) \delta_{ab}, \\ \hat{H}_5 &= \hat{G}_{(5)} + \hat{*}\hat{G}_{(5)}, \quad \hat{G}_{(5)} = 4ge^{(\alpha+20)\phi} \epsilon_5 - \frac{1}{2g} e^{(\gamma+4)\phi} *F^I \wedge dK^I. \end{aligned} \quad (2.2)$$

Here  $K^{Ia} = K^{Im} e_m^a$  are the orthonormal components of the Killing vectors on the internal space. The ansatz (2.2) is that used in [15, 16], except that we have now introduced a breathing-mode scalar  $\phi$ . The factors in the metric ansatz are chosen to yield an Einstein frame reduction, and also serve to normalize the breathing mode. Once this normalization is fixed, we have introduced constants  $\alpha$  and  $\gamma$  parameterising the breathing-mode dependence in the five-form ansatz. (The offsets are chosen for latter convenience.)

We first consider the  $\hat{H}_{(5)}$  equation of motion. At the linearized level in gauge fields, this separates into  $d\hat{G}_{(5)} = 0$  and  $d\hat{*}\hat{G}_{(5)} = 0$ . The former gives simply  $d(e^{(\gamma+4)\phi} *F^I) = 0$ , which is the linearization of the lower-dimensional Yang-Mills equation of motion. For  $d\hat{*}\hat{G}_{(5)} = 0$  we first compute

$$\hat{*}\hat{G}_{(5)} = 4ge^{\alpha\phi} (\omega_{(5)} - A^I \wedge *K^I) + \frac{1}{2g} e^{\gamma\phi} F^I \wedge *dK + \dots, \quad (2.3)$$

where we have only kept terms linear in the gauge fields, and where  $\omega_{(5)}$  is the volume form on the internal manifold. This allows us to write

$$\begin{aligned} d\hat{*}\hat{G}_{(5)} &= 4gde^{\alpha\phi} \wedge (\omega_{(5)} - A^I \wedge *K^I) + \frac{1}{2g} d(e^{\gamma\phi} F^I) \wedge *dK \\ &\quad + 4ge^{\alpha\phi} A^I \wedge d*K^I + \frac{1}{2g} e^{\gamma\phi} F^I \wedge (d*dK^I - 8g^2 e^{(\alpha-\gamma)\phi} *K^I). \end{aligned} \quad (2.4)$$

Using the Killing vector identities

$$d*K = 0, \quad d*dK - 2\Lambda *K = 0, \quad (2.5)$$

which follow from the Killing equation  $\nabla_m K_n^I + \nabla_n K_m^I = 0$  and the Einstein condition  $R_{mn} = \Lambda g_{mn}$ , we see that

$$d\hat{*}\hat{G}_{(5)} = 4gde^{\alpha\phi} \wedge (\omega_{(5)} - A^I \wedge *K^I) + \frac{1}{2g} de^{\gamma\phi} \wedge F^I \wedge *dK + 4ge^{\gamma\phi} (1 - e^{(\alpha-\gamma)\phi}) F^I \wedge *K^I, \quad (2.6)$$

where we have made use of the linearised Bianchi identity  $dF^I = 0$  and where we have taken  $\Lambda = 4g^2$ .

Demanding that  $d\hat{*}\hat{G}_{(5)} = 0$  requires the independent vanishing of terms with different tensor structure in (2.6). Starting with  $d\phi \wedge \omega_{(5)}$ , we see that  $\alpha = 0$ . After this, it is easy

to see that vanishing of the remaining terms demands  $\gamma = 0$ . Thus we conclude that  $\hat{*}\hat{G}_{(5)}$  cannot have any dependence on the breathing mode  $\phi$  at all. Furthermore, this conclusion would be unchanged even if we had allowed for more complicated functional dependence on  $\phi^1$ .

Having examined the equation of motion for  $\hat{G}_{(5)}$ , we may now turn to the lower-dimensional components of the IIB Einstein equation. This in particular allows us to examine the stress tensor associated with the Yang-Mills fields  $F^I$ . The reduction of the ten-dimensional Ricci tensor corresponding to (2.1) yields

$$\begin{aligned}\hat{R}_{\alpha\beta} &= e^{-5\phi}(R_{\alpha\beta} - \frac{5}{2}\eta_{\alpha\beta}\square\phi - 30\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}e^{-8\phi}K^{Ia}K_a^JF_{\alpha\gamma}^IF_\beta^{J\gamma}), \\ \hat{R}_{ab} &= e^{-5\phi}(e^{8\phi}R_{ab} + \frac{3}{2}\delta_{ab}\square\phi + \frac{1}{4}e^{-8\phi}K_a^IK_b^JF_{\alpha\beta}^IF^{J\alpha\beta}), \\ \hat{R}_{\alpha b} &= -\frac{1}{2}e^{-9\phi}K_b^I(D_\beta F_\alpha^{I\beta} - 8F_\alpha^{I\beta}\partial_\beta\phi),\end{aligned}\tag{2.7}$$

while

$$\hat{H}_{\alpha NPQR}\hat{H}_\beta^{NPQR} = -384g^2e^{15\phi}\eta_{\alpha\beta} + \frac{24}{g^2}e^{-\phi}\nabla_aK_b^I\nabla^aK^{Ib}(F_{\alpha\gamma}^IF_\beta^{J\gamma} - \frac{1}{4}\eta_{\alpha\beta}F_{\gamma\delta}^IF^{J\gamma\delta}).\tag{2.8}$$

Putting everything together yields the lower-dimensional Einstein equation

$$\begin{aligned}R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R &= 2g^2(5e^{8\phi} - 2e^{20\phi})\eta_{\alpha\beta} + 30(\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}\eta_{\alpha\beta}\partial_\gamma\phi\partial^\gamma\phi) \\ &\quad + \frac{1}{2}e^{-8\phi}(F_{\alpha\gamma}^IF_\beta^{J\gamma} - \frac{1}{4}\eta_{\alpha\beta}F_{\gamma\delta}^IF^{J\gamma\delta})Y^{IJ},\end{aligned}\tag{2.9}$$

where

$$Y^{IJ} = K^{Ia}K_a^J + \frac{1}{2g^2}e^{12\phi}\nabla_aK_b^I\nabla^aK^{Jb}\tag{2.10}$$

is constructed out of the 15 Killing vectors on the internal manifold. The first term in  $Y^{IJ}$  comes from the metric reduction, while the second arises through the five-form ansatz. Note that, at this order, we have ignored the coupling of the Yang-Mills fields to the non-breathing mode scalars. Although the full Einstein equation will involve these scalars, demonstrating the inconsistency of turning on a breathing mode will not require knowledge of these scalar couplings.

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<sup>1</sup>At first sight, it might seem that our conclusion that  $\alpha$  must vanish was an artefact of our having neglected to include any term in the ansatz for  $\hat{G}_{(5)}$  that involved  $d\phi$ , which might then have been able to cancel terms such as  $d\phi \wedge \omega_{(5)}$  arising in  $d\hat{*}\hat{G}_{(5)} = 0$ . However, such an addition to the ansatz is not possible. In order to cancel the  $d\phi \wedge \omega_{(5)}$  term in (2.6), it would be necessary to include a term of the form  $4\alpha ge^{\alpha\phi}d\phi \wedge \omega_{(4)}$  in (2.3), where  $d\omega_{(4)} = \omega_{(5)} + \dots$ . However, the required 4-form  $\omega_{(4)}$  on  $S^5$  is not globally defined, and thus such a term cannot appear in the ansatz. See section 3 for an extended discussion of this point.



Consistency of the Kaluza-Klein truncation demands that  $Y^{IJ}$  be independent of the internal coordinates  $y^m$ . Even in the absence of a breathing mode, this is a highly non-trivial condition, as it involves the cancellation of  $y^m$  dependence through the interplay of two terms. Specifically, for the  $SO(6)$  Killing vectors of  $S^5$  we have that  $K^{Ia} K_a^J$  is a  $y$ -dependent  $15 \times 15$  matrix of rank 5, while  $(\nabla_a K_b^I)(\nabla^a K^{Jb})$  is a  $y$ -dependent  $15 \times 15$  matrix of rank 10, but in their sum, with precisely the  $(2g^2)^{-1}$  coefficient in the second term, the  $y^m$  dependence cancels and the result is a constant matrix, of rank 15. By choosing the basis and normalisation for the Killing vectors appropriately, this constant rank-15 matrix can be chosen to be proportional to  $\delta^{IJ}$ .

If the breathing mode  $\phi$  is now turned on, it is evident that the previous consistency that required the precise balancing of the two Killing vector expressions in (2.10) will now fail. One way to express this is that consistency then demands a condition of the form

$$Y^{IJ} = \beta(\phi)\delta^{IJ}, \quad (2.11)$$

where  $\beta(\phi)$  is a function of  $\phi$  alone, and not of the internal  $y^m$  coordinates. Given the consistency of the truncation in the absence of a breathing mode, so that  $Y^{IJ} = \beta(0)\delta^{IJ}$  holds when  $\phi = 0$ , we may rewrite the above condition as

$$(1 - e^{12\phi})K^{Ia}K_a^J = \left(\beta(\phi) - e^{12\phi}\beta(0)\right)\delta^{IJ}. \quad (2.12)$$

This condition is clearly violated in general whenever the breathing mode is active, as the left hand side depends on  $y^m$ , while the right hand side does not. We have thus shown that it is inconsistent to perform a breathing-mode reduction of IIB supergravity on  $S^5$  that entails the coupling of the  $\mathcal{N} = 8$  gauged  $SO(6)$  supergravity to a breathing-mode supermultiplet.

Note that the consistent breathing-mode Sasaki-Einstein reductions [9, 10, 11, 12, 13] involve a single Abelian graviphoton corresponding to the gauging of the preferred  $U(1)$  isometry of the Sasaki-Einstein space. In the absence of a breathing mode, the consistency of such reductions involving  $U(1)$  bundled over a Kähler-Einstein base was studied in [16]. A convenient manipulation of (2.10) yields

$$Y^{IJ} = (1 + 2e^{12\phi})K^{Ia}K_a^J + \frac{1}{4g^2}e^{12\phi}\square(K^{Ia}K_a^J). \quad (2.13)$$

The  $U(1)$  graviphoton is associated with a Killing vector of constant length on  $S^5$ , which corresponds to taking  $K^{Ia} = \delta^{a9}$  (assuming 9 to be the  $U(1)$  fiber direction), in which case the second term above vanishes. The result is  $Y = 1 + 2e^{12\phi}$ , which yields a consistent

truncation with a lower dimensional Lagrangian of the form [10, 11, 12, 13]

$$\mathcal{L}_5 = R * 1 - \frac{1}{2}(1 + 2e^{12\phi})F_2 \wedge *F_2 + \dots \quad (2.14)$$

Finally, while we are primarily focused on supersymmetric breathing mode reductions, the condition (2.11) is universal, regardless of supersymmetry. We have thus shown that turning on a breathing mode is incompatible with retaining the non-Abelian gauge bosons corresponding to the gauging of a non-trivial isometry of the internal space. This result is of course compatible with the known breathing-mode reductions that either throw out the non-Abelian gauge bosons through a non-supersymmetric truncation [17], or work in an  $\mathcal{N} = 2$  context with only a single Abelian graviphoton [9, 10, 11, 12, 13].

### 3 An Explicit Example

In the previous section, we gave a general discussion that demonstrated the inconsistency of including the breathing mode in the  $S^5$  reduction of type IIB supergravity with the  $SO(6)$  gauge bosons. In this section, we shall consider a simplified sector within this analysis, in which only a  $U(1)$  subgroup of the  $SO(6)$  gauge fields is active. This will allow us to present completely explicit reduction ansätze, and to elaborate on some of the issues encountered in the previous section. We should emphasise that the  $U(1)$  subgroup that we shall consider here is *not* the one associated with the graviphoton considered in [9, 10, 11, 12, 13], and our discussion in this section will show that the inconsistency of including the breathing mode in an  $S^5$  reduction can be exhibited even with just a single suitably chosen  $U(1)$  gauge field turned on.

We take as our starting point the consistent  $S^5$  reduction of type IIB supergravity that yields five-dimensional  $\mathcal{N} = 4$  gauged  $SU(2) \times U(1)$  supergravity, which was studied in [18]. This is a consistent truncation of the full  $\mathcal{N} = 8$  gauged  $SO(6)$  reduction.

The five-dimensional bosonic fields of the consistent reduction exhibited in [18] comprised the metric; the gauge fields of  $SU(2) \times U(1)$ ; a single scalar field  $X$ ; and a complex 2-form potential. For the present purposes, we may consistently set the  $SU(2)$  gauge fields and the complex 2-form potential to zero. The only ten-dimensional fields that are non-vanishing for this simplified reduction ansatz are the metric  $\hat{g}_{MN}$  and the self-dual 5-form  $\hat{H}_{(5)}$ , satisfying the equations of motion (2.1). From [18], the consistent reduction ansatz is

then given by<sup>2</sup>

$$d\hat{s}_{10}^2 = \Delta^{1/2} ds_5^2 + g^{-2} \left( X \Delta^{1/2} d\xi^2 + X^2 \Delta^{-1/2} s^2 (d\tau - g A_{(1)})^2 + \Delta^{-1/2} X^{-1} c^2 d\Omega_3^2 \right), \quad (3.1)$$

$$\hat{G}_{(5)} = 2gU \epsilon_{(5)} - \frac{3sc}{g} X^{-1} *dX \wedge d\xi - \frac{sc}{g^2} X^4 *F_{(2)} \wedge d\xi \wedge \omega, \quad (3.2)$$

where

$$\begin{aligned} \Delta &= X^{-2} s^2 + X c^2, & U &= X^2 c^2 + X^{-1} s^2 + X^{-1}, & s &\equiv \sin \xi, & c &\equiv \cos \xi, \\ \omega &= d\tau - g A_{(1)}, & F_{(2)} &= dA_{(1)}, \end{aligned} \quad (3.3)$$

and the self-dual 5-form is written as  $\hat{H}_{(5)} = \hat{G}_{(5)} + \hat{*}\hat{G}_{(5)}$ . Note that  $d\Omega_3^2$  denotes the metric on the unit 3-sphere. It is useful also to record that the ten-dimensional dual of  $\hat{G}_{(5)}$  is

$$\hat{*}\hat{G}_{(5)} = -\frac{2sc^3}{g^4} U \Delta^{-2} d\xi \wedge \omega \wedge \Omega_3 + \frac{3s^2 c^4}{g^4} X^{-2} \Delta^{-2} dX \wedge \omega \Omega_3 - \frac{c^4}{g^3} X \Delta^{-1} F_{(2)} \wedge \Omega_{(3)}, \quad (3.4)$$

where  $\Omega_{(3)}$  is the volume form of the unit 3-sphere metric  $d\Omega_3^2$ .

Substituting the ansatz into the ten-dimensional equations of motion, one finds that the resulting five-dimensional equations can be derived from the Lagrangian [18]

$$\mathcal{L}_5 = R * \mathbb{1} - 3X^{-2} *dX \wedge dX - \frac{1}{2} X^4 *F_{(2)} \wedge F_{(2)} + 4g^2 (X^2 + 2X^{-2}) * \mathbb{1}. \quad (3.5)$$

Note from (3.2) and (3.4) that the equation of motion for the self-dual 5-form,  $d\hat{H} = 0$ , has the independent consequences that  $d\hat{G}_{(5)} = 0$  and  $d\hat{*}\hat{G}_{(5)} = 0$ .

We now investigate the effect of adding the breathing-mode scalar  $\phi$ . In the metric ansatz, this will modify (3.1) as in (2.2) to give

$$d\hat{s}_{10}^2 = e^{5\phi} \Delta^{1/2} ds_5^2 + g^{-2} e^{-3\phi} \left( X \Delta^{1/2} d\xi^2 + X^2 \Delta^{-1/2} s^2 (d\tau - g A_{(1)})^2 + \Delta^{-1/2} X^{-1} c^2 d\Omega_3^2 \right). \quad (3.6)$$

In the ansatz for the 5-form, it is easiest first to consider  $\hat{*}\hat{G}_{(5)}$ , for which we write

$$\hat{*}\hat{G}_{(5)} = -\frac{2sc^3}{g^4} e^{\alpha\phi} U \Delta^{-2} d\xi \wedge \omega \wedge \Omega_3 + \frac{3s^2 c^4}{g^4} e^{\beta\phi} (X \Delta)^{-2} dX \wedge \omega \wedge \Omega_3 - \frac{c^4}{g^3} e^{\gamma\phi} X \Delta^{-1} F_{(2)} \wedge \Omega_{(3)}, \quad (3.7)$$

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<sup>2</sup>Note that the Killing vector associated with the  $U(1)$  gauge field we are considering here is  $\partial/\partial\tau$ , which has length  $X \Delta^{-1/4} g^{-1} s^{-1}$ , and so it is not constant on  $S^5$  (even if  $X = 1$ ). This contrasts with the  $U(1)$  graviphoton gauge field retained in the consistent reduction considered in [9, 10, 11, 12, 13], which is associated with the Killing vector describing translations along the  $U(1)$  fibers and which does have constant length.

where  $\alpha$ ,  $\beta$  and  $\gamma$  are constants to be determined. The ansatz for  $\hat{G}_{(5)}$  itself can now easily be calculated, and we find

$$\hat{G}_{(5)} = 2gU e^{(\alpha+20)\phi} \epsilon_{(5)} - \frac{3sc}{g} e^{(\beta+12)\phi} X^{-1} *dX \wedge d\xi - \frac{sc}{g^2} e^{(\gamma+4)\phi} X^4 *F_{(2)} \wedge d\xi \wedge \omega, \quad (3.8)$$

(We postpone until later in this section the discussion of the possible addition of terms involving  $d\phi$  in the ansatz.)

As in the case without the breathing mode, the ten-dimensional equation  $d\hat{H}_{(5)} = 0$  leads to the separate equations  $d\hat{G}_{(5)} = 0$  and  $d*\hat{G}_{(5)} = 0$ . Looking at the various independent 6-forms in  $d*\hat{G}_{(5)} = 0$ , we find from the coefficients of  $dX \wedge d\xi \wedge \omega \wedge \Omega_3$ ,  $dX \wedge F_{(2)} \wedge \Omega_3$ ,  $d\xi \wedge F_{(2)} \wedge \Omega_3$  and  $d\phi \wedge d\xi \wedge \omega \wedge \Omega_3$  respectively that  $\alpha = \beta$ ,  $\beta = \gamma$ ,  $\alpha = \gamma$ , and  $\alpha = 0$ . Thus we conclude that  $\alpha = \beta = \gamma = 0$ .

Our interest is to look for an unambiguous signal of inconsistency (or consistency) in the putative reduction process in the presence of the breathing-mode scalar. To this end, we shall focus on a specific class of terms coming from the reduction procedure, namely those contributions of the form

$$R_{\alpha\beta} \sim F_{\alpha\gamma} F_{\beta}{}^{\gamma} + \dots \quad (3.9)$$

in the lower-dimensional Einstein equation. These terms quadratic in the  $U(1)$  gauge field come from two sources; one being from the ansatz (3.6) for the reduction of the ten-dimensional metric, and the other being from the  $F_{(2)}$  terms in the ansatz for  $\hat{H}_{(5)}$ . In [18], the ten-dimensional Ricci tensor is calculated for the reduction ansatz (3.1) without the breathing mode. After straightforward calculations, we find that for the metric reduction ansatz (3.6) that includes the breathing mode, the contribution to the lower-dimensional components of the higher-dimensional Ricci tensor of the form  $F_{\alpha\gamma} F_{\beta}{}^{\gamma}$  is

$$\hat{R}_{\alpha\beta} = \Delta^{-1/2} e^{-5\phi} R_{\alpha\beta} - \frac{1}{2} s^2 X^2 \Delta^{-3/2} e^{-13\phi} F_{\alpha\gamma} F_{\beta}{}^{\gamma} + \dots \quad (3.10)$$

From the ansatz (3.7) and (3.8) (with, as we then learned,  $\alpha = \beta = \gamma = 0$ ), we find the contribution

$$\hat{H}_{\alpha BCDE} \hat{H}_{\beta}{}^{BCDE} = 48c^2 e^{-\phi} X^5 \Delta^{-3/2} F_{\alpha\gamma} F_{\beta}{}^{\gamma} + \dots \quad (3.11)$$

Thus, from the ten-dimensional equations of motion (2.1) we find

$$R_{\alpha\beta} = \frac{1}{2} X^4 \Delta^{-1} e^{4\phi} (X^{-2} s^2 e^{-12\phi} + X c^2) F_{\alpha\gamma} F_{\beta}{}^{\gamma} + \dots \quad (3.12)$$

It should be emphasised that we have focused just on this specific type of structure, namely terms proportional to  $F_{\alpha\gamma} F_{\beta}{}^{\gamma}$ , and *no other terms that we have omitted could alter the form*

of these contributions. It is clear, therefore, that we have run into an inconsistency, since the left-hand side of (3.12) depends only on the lower-dimensional coordinates, whilst the right-hand side depends also on the coordinate  $\xi$  on the internal five-sphere. (Recall that  $s = \sin \xi$  and  $c = \cos \xi$ .) This is a classic example of what can go wrong in an inconsistent truncation.

It should be noted that if we had not introduced the breathing mode (*i.e.* if we had considered just the reduction investigated in [18]), then all would have been well. With  $\phi = 0$ , the  $\xi$  dependence of the two terms on the right-hand side would cancel, since then the factor in parentheses is just  $(X^{-2}s^2 + Xc^2)$ , which equals  $\Delta$ , thus cancelling the ( $\xi$ -dependent)  $\Delta^{-1}$  factor.

We now return to the question of whether terms involving  $d\phi$  could have been introduced in the ansatz for  $\hat{G}_{(5)}$ , thereby circumventing the conclusion arrived at in the discussion below (3.8), namely that  $\alpha = \beta = \gamma = 0$ . After some experimentation, we find that if we add the term

$$\hat{*}\hat{G}_{(5)}^{\text{extra}} = \frac{\alpha c^4}{g^4} e^{\alpha\phi} X \Delta^{-1} d\phi \wedge \omega \wedge \Omega_3 \quad (3.13)$$

to (3.7), then we satisfy  $d\hat{*}\hat{G}_{(5)} = 0$  identically, provided only that we impose  $\alpha = \beta = \gamma$ . In fact, we then have that with this extra term included,

$$\hat{*}\hat{G}_{(5)} = d\left(\frac{c^4}{g^4} e^{\alpha\phi} X \Delta^{-1} \omega \wedge \Omega_{(3)}\right). \quad (3.14)$$

With the apparent freedom now to allow  $\alpha \neq 0$ , which would introduce an extra factor of  $e^{2\alpha\phi}$  in (3.11) and in the second term in (3.12), it might seem that we could now restore consistency in the Einstein equations by choosing  $\alpha = -6$  so that the  $\xi$ -dependence on the right-hand side of (3.12) cancels, yielding

$$R_{\alpha\beta} = \frac{1}{2} X^4 e^{-8\phi} F_{\alpha\gamma} F_{\beta}{}^{\gamma} + \dots \quad (3.15)$$

However, the extra term (3.13) that needed to be added is actually singular, as may be seen by calculating the associated vielbein components, which are

$$(\hat{*}\hat{G}_{(5)}^{\text{extra}})_{\alpha 6123} = \frac{\alpha c}{s} \Delta^{-5/4} e^{(\alpha+7/2)\phi} X^{-7/4} \partial_{\alpha}\phi, \quad (3.16)$$

(where the 1, 2 and 3 directions are on  $S^3$ , and 6 is along the  $\omega$  direction). Thus the vielbein components blow up at the north and south poles of the 5-sphere, where  $\sin \xi = 0$ . This is a concrete realisation of the point alluded to in section 2, that the needed extra term in the ansatz that might have allowed  $\alpha$  to be non-zero is proportional to  $d\phi \wedge \omega_{(4)}$ , where  $\omega_{(4)}$  is a (necessarily singular) 4-form whose exterior derivative yields the volume form of  $S^5$ .

There are in fact even worse problems associated with the introduction of the extra term (3.13). To see these, it suffices to set  $X = 1$  and  $A_{(1)} = 0$ , and focus on the contributions of the breathing-mode scalar in the lower-dimensional components of the ten-dimensional Einstein equations. We find that these are given by

$$R_{\alpha\beta} = \frac{5}{8}\square\phi g_{\alpha\beta} + 30\partial_\alpha\phi\partial_\beta\phi + \frac{\alpha^2 c^2}{4s^2} e^{(2\alpha+12)\phi} (2\partial_\alpha\phi\partial_\beta\phi - (\partial\phi)^2 g_{\alpha\beta}), \quad (3.17)$$

where the terms on the right-hand side that are proportional to  $\alpha^2$  are the contribution from (3.13). The  $\xi$ -dependence of these terms implies that we have a classic example of an inconsistent reduction. This problem would not, of course, be remedied by restoring the  $X$  and  $A_{(1)}$  fields that we omitted in this simplified discussion.

The upshot of this discussion is that the conclusion reached in the discussion following eqn (3.8) is robust; the constants  $\alpha$ ,  $\beta$  and  $\gamma$  should all be zero, and hence from eqn (3.12) we have indeed demonstrated the inconsistency of retaining the breathing mode as well as the  $U(1)$  gauge field of this truncation.

## 4 Conclusions

In this paper we have investigated the idea, motivated by the conjecture made in [12], that it might be possible to augment the usual consistent  $S^5$  reduction of type IIB supergravity to five-dimensional massless  $\mathcal{N} = 8$  gauged  $SO(6)$  supergravity, by including as well the massive supermultiplet associated with the breathing-mode scalar. Our conclusion is that such an extended reduction is, unfortunately, not a consistent one. In fact our result is somewhat stronger, and quite independent of supersymmetry. Namely, we have shown that the consistency of the reduction to the massless bosonic sector (including the gauge fields of  $SO(6)$ ) is destroyed as soon as one tries to augment the ansatz to include the breathing-mode scalar.

An essential ingredient in our argument is that one can focus on certain subsets of interactions within the full reduction procedure, which would not in themselves provide fully complete and consistent reductions, provided one is careful to extract only those conclusions that are insensitive to the presence or absence of the omitted terms. In this spirit, we extended a consistency test that was first developed long ago to probe the validity of including the non-Abelian gauge bosons in the  $S^7$  reduction of  $D = 11$  supergravity and the  $S^5$  reduction of type IIB supergravity. The consistency of these reductions depends on highly non-trivial conspiracies involving the details of the higher-dimensional theories and the properties of the Killing vectors on the compactifying spheres. In our extension of this

analysis, we showed in the case of the  $S^5$  reduction of type IIB supergravity, these delicate conspiracies are destroyed if the breathing mode is introduced as well.

The conclusion is that neither the breathing mode, nor its entire associated massive supermultiplet, can be included in any enlargement of the existing consistent reduction of type IIB supergravity to five-dimensional  $\mathcal{N} = 8$  gauged  $SO(6)$  supergravity. Analogous arguments would lead to a similar conclusion for the  $S^7$  reduction of eleven-dimensional supergravity.

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